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Quantum superimposing multiple cloning-cum-complementing machine

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Abstract
We prove that the nonorthogonal states randomly selected from a set can evolve into a linear superposition of multiple original and orthogonal complementing states with failure branch if and only if the input states are linearly independent. The results for a single-input state are also generalized into the case of several copies of an input state.

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1. Introduction

Due to the linearity and unitarity of quantum theory, there are some restrictions and limitations on manipulations with quantum information: for example, the quantum no-cloning theorem [1], which prohibits an arbitrary unknown state from being perfectly cloned; the quantum no-deleting theorem [2, 3], which proves the impossibility of perfectly deleting an unknown state against a copy; the quantum no-disentangling theorem [4, 5], which asserts that there cannot exist the exact disentanglement machine for any unknown quantum state; the quantum no-flipping theorem [6–10], which says that exact flipping of any unknown qubit is not possible, i.e., there exists no perfect flipper which can operate on any unknown qubit state $|\psi\rangle$ resulting in the orthogonal state $|\psi^{\perp}\rangle$; the quantum no-complementing theorem [6, 7, 11], which states the impossibility of producing an exact orthogonal complementing state along with the original starting from a single copy. In other words, the quantum no-complementing theorem indeed reflects the quantum no-anti-cloning properly. Though the perfect operations mentioned above are not possible, one may realize these impossible operations either in a probabilistic but an exact or deterministic but inaccurate way [7, 11–21]. For example, probabilistic cloning was first proposed by Duan and Guo [13]; other authors developed it from a different point of view [16, 19, 21]. With the great advances in the quantum information theory, understanding the limits of the manipulations that we can perform on quantum information becomes more and more important. These limits tell us what we can do with the information contained in unknown states and what we cannot. All of them unveil the unique properties of quantum information from different aspects.

In spite of the intensive work on manipulation and extraction of quantum information, surprising effects are continuously being discovered. For instance, in any physical situation the cloning operation and complementing transformation (i.e. the universal NOT gate) are deeply interconnected [7]; these two processes are always realized contextually and their optimal fidelity is directly related [22, 23]. Another interesting related observation is that the two anti-parallel spin states contain more information than two parallel spin states [6], i.e., one can measure the spin direction of $|\psi\rangle$ with better fidelity when two qubits are in anti-parallel spin state $|\psi, \psi^{\perp}\rangle$ than in parallel one $|\psi, \psi\rangle$. Therefore, it is important to consider a quantum complementing machine (i.e. an anti-cloning device) and a spin-flip machine [6]. Actually, a probabilistic quantum anti-cloning machine and a probabilistic quantum flipping machine were proposed in [11, 12]. Two machines respectively demonstrated that the nonorthogonal state secretly chosen from a set can be faithfully anti-cloned into an orthogonal complementing state along with the original or flipped into the orthogonal state with certain probabilities if and only if the states are linearly independent. We note that in the above two processes only a single state is produced probabilistically, i.e., $|\psi\rangle \to |\psi\rangle|\psi^{\perp}\rangle$ or $|\psi\rangle \to |\psi^{\perp}\rangle$ is merely generated with certain probabilities. However, in quantum world, one can have linear superposition of all possibilities with appropriate probabilities [16, 24–26]. If a real quantum cloning-cum-complementing machine existed, it would take advantage of this basic quantum property and produce simultaneously $|\psi\rangle \to |\psi^{\perp}\rangle^{\otimes M}$, $|\psi\rangle \to |\psi\rangle|\psi^{\perp}\rangle^{\otimes(M-1)}$, $|\psi\rangle \to |\psi\rangle^{\otimes 2}|\psi^{\perp}\rangle^{\otimes(M-2)}, \ldots\ldots, |\psi\rangle \to |\psi\rangle^{\otimes M}$, where M is a positive integer. Motivated by these, it is naturally desirable to ask whether there can exist a quantum machine which takes an unknown quantum state as an input state and produces an output state which will be in a linear superposition of all possible multiple original and orthogonal complementing states. The answer is positive.

In this paper, we propose a quantum superimposing multiple cloning-cum-complementing machine in section 2. We prove that the nonorthogonal states randomly selected from a set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ can evolve into a linear superposition of multiple original and orthogonal complementing states with the failure branch described by a composite state (independent of the input state) by a unitary evolution together with a measurement if and only if the states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle$ are linearly independent. And we derive a bound on the success probabilities of our machine. The generality of this machine, and the generalization to the case of several input copies of a state are discussed in section 3. We show that the probabilistic flipping machine [11, 12], the probabilistic anti-cloning machine [11], the probabilistic cloning machine [13] and the probabilistic multiple anti-cloning machine are special cases of our multiple cloning-cum-complementing machine; the 'linear superposition of multiple copies and complements of orthogonal states' theorem and 'no-superposition of multiple clones-cum-complements' theorem can also be obtained from our machine. Our summary is presented in section 4.

2. Quantum superimposing multiple cloning-cum-complementing machine

We first consider under what conditions the states randomly selected from the set can be evolved into a linear superposition of multiple original and orthogonal complementing states together with failure branch by a unitary evolution and a measurement. This is answered by theorem 1.

Theorem 1. *There exists a unitary operator U such that for any unknown nonorthogonal state chosen from a set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ the machine can produce a linear superposition of multiple clones-cum-complements together with failure branch given by*

$$U(|\psi_i\rangle|\Sigma\rangle|P\rangle) = \sum_{n=0}^{M}\sqrt{p_n^{(i)}}|\psi_i\rangle^{\otimes n}\left|\psi_i^{\perp}\right\rangle^{\otimes(M-n)}|P_n\rangle + \sum_{l=M+1}^{N_C}\sqrt{f_l^{(i)}}|\phi_l\rangle_{AB}|P_l\rangle,$$
$$(i = 1, 2, \ldots, k), \tag{1}$$

if and only if the states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle$ are linearly independent. The machine is named as the quantum multiple cloning-cum-complementing machine.

In equation (1), the unknown input state $|\psi_i\rangle$ from a set S belongs to a Hilbert space $H_A = C^{N_A}$; $|\Sigma\rangle$ is the state of the ancillary system B belonging to a Hilbert space H_B of dimension $N_B = N_A^{M-1}$, where $(M-1)$ is the total number of blank states each having dimension N_A; $\left|\psi_i^{\perp}\right\rangle$ is the orthogonal complementing state of $|\psi_i\rangle$. If equation (1) holds, the probe P is measured after the evolution. The output states are reserved if and only if the measurement results of the probe are $|P_n\rangle$ $(n = 0, 1, \ldots, M)$. So p_n^i is the success probability for the ith input state $|\psi_i\rangle$ to produce n exact copies $|\psi_i\rangle^{\otimes n}$ and $(M-n)$ exact complementing orthogonal states $\left|\psi_i^{\perp}\right\rangle^{\otimes(M-n)}$; $f_l^{(i)}$ is the failure probability for the ith input state to remain in the lth failure component. $\sum_l\sqrt{f_l^{(i)}}|\phi_l\rangle_{AB}|P_l\rangle$ represents the failure branch. The states $|\phi_l\rangle_{AB}$ are normalized states of the composite system AB and they are not necessarily orthogonal. $|P\rangle, |P_0\rangle, |P_1\rangle, |P_2\rangle, \ldots, \left|P_{N_C}\right\rangle$ are the orthonormal basis states of the probing device with $N_C > M$. To prove that equation (1) holds with a positive probability p_n^i, we introduce the following lemmas.

Lemma 1. *If the set $S_1 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent, then the set $S = \left\{|\psi_1\rangle^{\otimes n}\left|\psi_1^{\perp}\right\rangle^{\otimes(M-n)}, |\psi_2\rangle^{\otimes n}\left|\psi_2^{\perp}\right\rangle^{\otimes(M-n)}, \ldots, |\psi_k\rangle^{\otimes n}\left|\psi_k^{\perp}\right\rangle^{\otimes(M-n)}\right\}$ is linearly independent, where M, n are positive integers and $M > n$.*

Proof. We use the 'negative approach' to prove lemma 1. Let us suppose that the set $S = \left\{|\psi_1\rangle^{\otimes n}\left|\psi_1^{\perp}\right\rangle^{\otimes(M-n)}, |\psi_2\rangle^{\otimes n}\left|\psi_2^{\perp}\right\rangle^{\otimes(M-n)}, \ldots, |\psi_k\rangle^{\otimes n}\left|\psi_k^{\perp}\right\rangle^{\otimes(M-n)}\right\}$ is linearly dependent. Then there exists

$$\sum_{i=1}^{k} c_i|\psi_i\rangle^{\otimes n}\left|\psi_i^{\perp}\right\rangle^{\otimes(M-n)} = 0 \tag{2}$$

with c_i $(i = 1, 2, \ldots, k)$ being not all zero. For an arbitrary unknown state $|\psi\rangle$ there exists $K|\psi\rangle = |\psi^{\perp}\rangle$, where K is an antiunitary operator which satisfies [7, 12]: (i) $\langle\psi|\varphi\rangle = \langle\psi'|\varphi'\rangle^*$, where $|\psi'\rangle = K|\psi\rangle$, $|\varphi'\rangle = K|\varphi\rangle$ and (ii) $K\sum_{i=1}^{n} c_i|i\rangle = \sum_{i=1}^{n} c_i^* K|i\rangle$, where $\{|i\rangle\}$ is an orthonormal basis of n-dimensional Hilbert space. Consequently, equation (2) is reduced to

$$K^{(M-n)}\sum_{i=1}^{k} c_i|\psi_i\rangle^{\otimes M} = 0 \qquad (M-n \text{ being an even number}), \tag{3a}$$

$$K^{(M-n)}\sum_{i=1}^{k} c_i^*|\psi_i\rangle^{\otimes M} = 0 \qquad (M-n \text{ being an odd number}). \tag{3b}$$

From equation (3) we obtain that $\sum_{i=1}^{k} c_i|\psi_i\rangle^{\otimes M} = 0$ or $\sum_{i=1}^{k} c_i^*|\psi_i\rangle^{\otimes M} = 0$ for each allowed positive integer $(M-n)$, which means that the set $S = \{|\psi_1\rangle^{\otimes M}, |\psi_2\rangle^{\otimes M}, \ldots, |\psi_k\rangle^{\otimes M}\}$ is linearly dependent.

However, it is easy to prove that the set $S = \{|\psi_1\rangle^{\otimes n}, |\psi_2\rangle^{\otimes n}, \ldots, |\psi_k\rangle^{\otimes n}\}$ $(n > 1)$ is linearly independent if the set $S_1 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent (see the appendix). Therefore, it is in contradiction with the result of the above assumption, so the set $S = \left\{|\psi_1\rangle^{\otimes n}\left|\psi_1^{\perp}\right\rangle^{\otimes(M-n)}, |\psi_2\rangle^{\otimes n}\left|\psi_2^{\perp}\right\rangle^{\otimes(M-n)}, \ldots, |\psi_k\rangle^{\otimes n}\left|\psi_k^{\perp}\right\rangle^{\otimes(M-n)}\right\}$ is linearly independent. □

Lemma 2. *If the set* $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ *is linearly independent then the matrix* $X = \left[\langle\psi_i|\psi_j\rangle^n\left\langle\psi_i^{\perp}\middle|\psi_j^{\perp}\right\rangle^{M-n}\right]$ *is positive definite, where* M, n *are positive integers and* $(M - n) \geqslant 0$.

Proof. For an arbitrary column vector $\alpha = \mathrm{col}(c_1, c_2, \ldots, c_k)$, the quadratic form $\alpha^+ X\alpha$ can be expressed as

$$\alpha^+ X\alpha = \sum_{i,j=1}^{k} c_i^* c_j X_{ij} = \langle\beta|\beta\rangle \tag{4}$$

where $|\beta\rangle = \sum_i c_i|\psi_i\rangle^{\otimes n}\left|\psi_i^{\perp}\right\rangle^{\otimes(M-n)}$. Since $|\psi_1\rangle, |\psi_2\rangle, \ldots$ and $|\psi_k\rangle$ are linearly independent, according to lemma 1, the summation state $|\beta\rangle$ does not reduce to zero for any k vector α, and its norm $\langle\beta|\beta\rangle$ is therefore always positive. By equation (4), we conclude that the matrix X is positive definite. □

Now we prove theorem 1 in two stages. First, we show that if such a machine described by equation (1) exists, then $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle$ are linearly independent. Consider an arbitrary state $|\psi\rangle = \sum_{i=1}^{k} c_i|\psi_i\rangle$. If we send this state through the machine defined by equation (1), the unitary evolution will yield

$$U(|\psi\rangle|\Sigma\rangle|P\rangle) = \sum_{n=0}^{M} \sqrt{p_n}|\psi\rangle^{\otimes n}|\psi^{\perp}\rangle^{\otimes(M-n)}|P_n\rangle + \sum_{l=M+1}^{N_C} \sqrt{f_l}|\phi_l\rangle_{AB}|P_l\rangle. \tag{5}$$

However, by linearity of the quantum theory each of $|\psi_i\rangle$ $(i = 1, 2, \ldots, k)$ will go through the transformation (1) and we obtain

$$U\left(\sum_{i=1}^{k} c_i|\psi_i\rangle|\Sigma\rangle|P\rangle\right) = \sum_{i=1}^{k} c_i \sum_{n=0}^{M} \sqrt{p_n^{(i)}}|\psi_i\rangle^{\otimes n}\left|\psi_i^{\perp}\right\rangle^{\otimes(M-n)}|P_n\rangle$$
$$+ \sum_{i=1}^{k} c_i \sum_{l=M+1}^{N_C} \sqrt{f_l^{(i)}}|\phi_l\rangle_{AB}|P_l\rangle. \tag{6}$$

From equations (5) and (6), if and only if $|\psi\rangle = |\psi_i\rangle$, the final states in equations (5) and (6) are the same; otherwise, the final states in equations (5) and (6) are different, which means that the quantum state $|\psi\rangle$ cannot exist in a linear superposition of all possible states. We know that if a set $\{|\psi\rangle, |\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ contains distinct vectors such that $|\psi\rangle$ is a linear combination of other $|\psi_i\rangle$'s, then the set is linearly dependent. Thus linearity prohibits us from producing linear superposition of multiple clones-cum-complements for the input states chosen from a linearly dependent set. Therefore, the unitary evolution (1) exists for any state randomly selected from S only if the set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent.

Conversely, we demonstrate that the linear independence of $\{|\psi_i\rangle\}$ $(i = 1, 2, \ldots, k)$ results in the existence of unitary operator U given by equation (1). If and only if there exists a unitary operator U satisfying equation (1), the inner product of equation (1) yields the equation

$$\langle\psi_i|\psi_j\rangle = \sum_{n=0}^{M}\sqrt{p_n^{(i)}}\langle\psi_i|\psi_j\rangle^n\langle\psi_i^{\perp}|\psi_j^{\perp}\rangle^{(M-n)}\sqrt{p_n^{(j)}} + \sum_{l=M+1}^{N_C}\sqrt{f_l^{(i)}f_l^{(j)}},$$
$$(i=1,2,\ldots,k \quad j=1,2,\ldots,k). \tag{7}$$

Equation (7) can be denoted by the $k\times k$ matrix equation

$$G^{(1)} = \sum_{n=0}^{M} A_n X A_n^{+} + \sum_{l=M+1}^{N_C} F_l, \tag{8}$$

where the matrices $G^{(1)} = [\langle\psi_i|\psi_j\rangle]$, $X = \left[\langle\psi_i|\psi_j\rangle^n\langle\psi_i^{\perp}|\psi_j^{\perp}\rangle^{M-n}\right]$, $A_n = A_n^{+} = \mathrm{diag}\left(\sqrt{p_n^{(1)}},\sqrt{p_n^{(2)}},\ldots,\sqrt{p_n^{(k)}}\right)$ and $F_l = \left[\sqrt{f_l^{(i)}f_l^{(j)}}\right]$.

By lemma 1 given in [13], if equation (8) holds, then there exists a unitary operator U satisfying equation (1). So we aim to prove that if the set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent, then equation (8) holds for a positive definite matrix A_n. If the nonorthogonal states $\{|\psi_i\rangle\}$ $(i=1,2,\ldots,k)$ are linearly independent, it can be shown that the matrix $G^{(1)}$ is positive definite [13] and the matrix X is positive definite too according to lemma 2. From continuity, for small enough but positive $p_n^{(i)}$ $(i=1,2,\ldots,k)$, the matrix $G^{(1)} - \sum_{n=0}^{M} A_n X A_n^{+}$ is also positive definite. Therefore, we can diagonalize the Hermitian matrix $G^{(1)} - \sum_{n=0}^{M} A_n X A_n^{+}$ by a suitable unitary operator V as follows:

$$V^{+}\left(G^{(1)} - \sum_{n=0}^{M} A_n X A_n^{+}\right)V = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_k), \tag{9}$$

where the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_k$ are positive real numbers. In equation (8), we can choose

$$F_l = V\,\mathrm{diag}\left(t_l^{(1)},t_l^{(2)},\ldots,t_l^{(k)}\right)V^{+} \tag{10}$$

such that

$$\sum_{l=M+1}^{N_C} t_l^{(i)} = \lambda_i \qquad (i=1,2,\ldots,k). \tag{11}$$

Equations (9)–(11) indicate that equation (8) is satisfied with a positive definite matrix A_n if the states are linearly independent. This completes the proof of theorem 1.

Now, we derive a bound on the success probabilities of this unitary transformation. Taking the inner product of two distinct states given by equation (1) and using relation $|\langle\psi_i|\psi_j\rangle| = \left|\langle\psi_i^{\perp}|\psi_j^{\perp}\rangle\right|$, we have

$$|\langle\psi_i|\psi_j\rangle| \leqslant \sum_{n=0}^{M}\sqrt{p_n^{(i)}}|\langle\psi_i|\psi_j\rangle|^M\sqrt{p_n^{(j)}} + \sum_{l=M+1}^{N_C}\sqrt{f_l^{(i)}f_l^{(j)}}. \tag{12}$$

By using the condition of normalization

$$\sum_{n=0}^{M} p_n^{(i)} + \sum_{l=M+1}^{N_C} f_l^{(i)} = 1, \tag{13}$$

we can obtain

$$\frac{1}{2}\sum_{n=0}^{M}\left(p_n^{(i)} + p_n^{(j)}\right) \leqslant (1 - |\langle\psi_i|\psi_j\rangle|)/(1 - |\langle\psi_i|\psi_j\rangle|^M). \tag{14}$$

The equality in (14) holds if and only if $\langle\psi_i|\psi_j\rangle = \langle\psi_i^{\perp}|\psi_j^{\perp}\rangle = |\langle\psi_i|\psi_j\rangle|$ and $p_n^{(i)} = p_n^{(j)}$.

The inequality (14) indicates that the sum of success probabilities of two distinct multiple cloning-cum-complementing states is always bounded by M and inner product of two corresponding input nonorthogonal states.

3. Discussion and generalization

Now we discuss the generality of our multiple cloning-cum-complementing machine.

If the states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle$ are orthogonal and all success probabilities $p_n^{(i)}$'s are nonzero, then the unitary evolution (1) allows us to have a 'linear superposition of multiple copies and complements of orthogonal states' theorem since the matrix equation (8) is always satisfied.

If all the failure probabilities $f_l^{(i)}$'s are zero, from the proof of theorem 1, we have that equation (8) cannot be satisfied with $p_n^{(i)} > 0$ for nonorthogonal states $|\psi_i\rangle$ $(i = 1, 2, \ldots, k)$. Therefore, we have a 'no-superposition of multiple clones-cum-complements' theorem.

If we take that $M = 1$, $p_0^{(i)}$ $(i = 1, 2, \ldots, k)$ are non-zero and all others are zero, then equation (1) reduces to

$$U(|\psi_i\rangle|P\rangle) = \sqrt{p_0^{(i)}}\,|\psi_i^{\perp}\rangle|P_0\rangle + \sum_l \sqrt{f_0^{(i)}}\,|\psi_l\rangle_{AB}|P_l\rangle, \tag{15}$$

which describes a probabilistic flipping machine discussed in [11, 12]. The bound on the success probabilities (14) reduces to $\frac{1}{2}\big(p_0^{(i)} + P_0^{(j)}\big) \leqslant 1$. When $\langle\psi_i|\psi_j\rangle = \big\langle\psi_i^{\perp}\big|\psi_j^{\perp}\big\rangle = |\langle\psi_i|\psi_j\rangle|$ and $p_0^{(i)} = p_0^{(j)}$, we reach the equality

$$p_0^{(i)} = p_0^{(j)} = 1. \tag{16}$$

It corresponds to the result [27, 28] that the qubits chosen from the polar great circle on the Bloch sphere can be flipped by the same unitary operator.

If we take that $M = 2$, $p_1^{(i)}$ $(i = 1, 2, \ldots, k)$ are non-zero and all others are zero, then equation (1) reduces to

$$U(|\psi_i\rangle|\Sigma\rangle|P\rangle) = \sqrt{p_1^{(i)}}\,|\psi_i\rangle\big|\psi_i^{\perp}\big\rangle|P_1\rangle + \sum_l \sqrt{f_1^{(i)}}\,|\phi_l\rangle_{AB}|P_l\rangle, \tag{17}$$

which describes the probabilistic anti-cloning machine discussed in [11]. In this case, our bound (14) reduces to

$$\tfrac{1}{2}\big(p_1^{(i)} + p_1^{(j)}\big) \leqslant 1/(1 + |\langle\psi_i|\psi_j\rangle|). \tag{18}$$

If we take that $M = 2$, $p_2^{(i)}$ $(i = 1, 2, \ldots, k)$ are non-zero and all others are zero, then equation (1) reduces to

$$U(|\psi_i\rangle|\Sigma\rangle|P\rangle) = \sqrt{p_2^{(i)}}\,|\psi_i\rangle|\psi_i\rangle|P_2\rangle + \sum_l \sqrt{f_l^{(i)}}\,|\phi_l\rangle_{AB}|P_l\rangle. \tag{19}$$

Our bound (14) reduces to $\frac{1}{2}\big(p_2^{(i)} + p_2^{(j)}\big) \leqslant 1/(1 + |\langle\psi_i|\psi_j\rangle|)$, which is the same as the Duan–Guo bound [13] for producing two clones in a probabilistic fashion.

If we take that M is an even number, $p_{M/2}^{(i)}$ $(i = 1, 2, \ldots, k)$ are non-zero and all others are zero, then equation (1) reduces to

$$U(|\psi_i\rangle|\Sigma\rangle|P\rangle) = \sqrt{p_{M/2}^{(i)}}\big(|\psi_i\rangle\big|\psi_i^{\perp}\big\rangle\big)^{\otimes(M/2)}|P_{M/2}\rangle + \sum_l \sqrt{f_l^{(i)}}\,|\phi_l\rangle_{AB}|P_l\rangle, \tag{20}$$

which can be regarded as the probabilistic multiple anti-cloning machine. Our bound (14) reduces to

$$\tfrac{1}{2}\big(p_{M/2}^{(i)} + p_{M/2}^{(j)}\big) \leqslant (1 - |\langle\psi_i|\psi_j\rangle|)/(1 - |\langle\psi_i|\psi_j\rangle|^M). \tag{21}$$

The above analyses show that our machine given in theorem 1 is general to some degree. Nevertheless, theorem 1 is limited to taking a single copy of $|\psi_i\rangle$ as an input state. Then, can $m(m$ is a positive integer and $m > 1)$ copies of nonorthogonal state $|\psi_i\rangle$ secretly chosen from the set S be evolved into a linear superposition of multiple original and orthogonal complementing states by a general unitary-reduction operation? The answer is 'yes' and the generalization of the superimposing multiple cloning-cum-complementing machine for a single-input copy $|\psi_i\rangle$ to that for m input copies $|\psi_i\rangle^{\otimes m}$ is straightforward; it can be completed by a way similar to the proof of theorem 1. For the sake of conciseness, we omit its proof. The results are displayed as follows.

Theorem 2. *For any unknown nonorthogonal state chosen from a set $S = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$, there exists a unitary operator U such that $|\psi_i\rangle^{\otimes m}$ can be evolved into a linear superposition of multiple clones-cum-complements together with the failure branch given by*

$$U(|\psi_i\rangle^{\otimes m}|\Sigma_m\rangle|P\rangle) = \sum_{n=0}^{M}\sqrt{p_n^{(i)}}|\psi_i\rangle^{\otimes n}\big|\psi_i^{\perp}\big\rangle^{\otimes(M-n)}|P_n\rangle + \sum_{l=M+1}^{N_C}\sqrt{f_l^{(i)}}|\Phi_l\rangle_{AB}|P_l\rangle,$$
$$(i = 1, 2, \ldots, k), \tag{22}$$

if and only if $|\psi_1\rangle^{\otimes m}, |\psi_2\rangle^{\otimes m}, \ldots, |\psi_k\rangle^{\otimes m}$ are linearly independent.

In equation (22), $|\Sigma_m\rangle$ is the state of the ancillary system B belonging to Hilbert space H_B of dimension $N_B = N_A^{M-m}$, where M, m are positive integers and $M > m$, $(M - m)$ is the total number of blank states each having dimension N_A. $p_n^{(i)}$ and $f_l^{(i)}$ denote the success and failure probabilities for the ith input state $|\psi_i\rangle^{\otimes m}$ to produce n exact copies $|\psi_i\rangle^{\otimes n}$ and $(M-n)$ exact orthogonal complementing states $\big|\psi_i^{\perp}\big\rangle^{\otimes(M-n)}$ respectively. The other quantities have the same meaning as explained above.

The bound on the success probabilities of our general machine (22) is

$$\frac{1}{2}\sum_{n=0}^{M}\big(p_n^{(i)} + p_n^{(j)}\big) \leqslant (1 - |\langle\psi_i|\psi_j\rangle|^m)/(1 - |\langle\psi_i|\psi_j\rangle|^M). \tag{23}$$

It is obvious that theorem 1 is a special case of theorem 2 with $m = 1$.

The quantum machine described by equation (22) takes m copies of $|\psi_i\rangle$ as an input state and produces an output state whose success branch is the linear superposition of n exact copies of $|\psi_i\rangle$ and $(M - n)$ exact copies of $\big|\psi_i^{\perp}\big\rangle$ and the failure branch is the superposition of composite states independent of the input state. In the case of $n \leqslant m$, some of the copies are flipped. In the case of $m < n \leqslant M$, the input states are cloned and flipped at the same time.

Finally, we note that our approach considers the spin-flipping, anti-cloning and cloning transformation and has a linear superposition of multiple clones-cum-complements. On the other hand, Qiu [12] analyzed some general probabilistic quantum cloning and deleting machines with a general unitary or antiunitary operator appearing on the right-hand side of the equation used to describe the machine model, but Qiu [12] did not include the linear superposition of all possible multiple states. Then, we may generalize our theorem 1 to the case that a general unitary or antiunitary operator appearing on the right-hand side of the equation described the machine model as a further task.

4. Summary

In conclusion, we have constructed a quantum superimposing multiple cloning-cum-complementing machine. We have proved that the nonorthogonal states randomly selected from a certain set S can evolve into a linear superposition of multiple original and orthogonal complementing states with failure branch by a unitary evolution together with a measurement if and only if the input states are linearly independent. We have derived a bound on the success probabilities of our machine. We have shown that the probabilistic flipping machine [11, 12], the probabilistic anti-cloning machine [11], the probabilistic cloning machine [13] and the probabilistic multiple anti-cloning machine are special cases of our machine; the 'linear superposition of multiple copies and complements of orthogonal states' theorem and 'no-superposition of multiple clones-cum-complements' theorem can also be obtained from our machine. These results for a single-input state have been generalized into the case of several copies of an input state. Our result may have potential applications in quantum information processing (such as quantum state engineering, anti-parallel storage of quantum information, etc) because it provides an intrinsic regularity of quantum states in the quantum computer. It also tells us how to control the success probability for the input states to produce multiple original and orthogonal complementing states in a desired way by using controllable operations. Moreover, we hope that our result can play a fundamental role in future understanding of quantum-information theory.

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Appendix

Now, we prove that the set $S = \{|\psi_1\rangle^{\otimes n}, |\psi_2\rangle^{\otimes n}, \ldots, |\psi_k\rangle^{\otimes n}\}$ $(n > 1)$ is linearly independent if the set $S_1 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent. The proof is as follows.

Proof. If the set $S_1 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent, let us suppose that $S = \{|\psi_1\rangle^{\otimes n}, |\psi_2\rangle^{\otimes n}, \ldots, |\psi_k\rangle^{\otimes n}\}$ is linearly dependent, i.e., there exists

$$\sum_i c_i |\psi_i\rangle^{\otimes n} = 0, \tag{A.1}$$

with c_i $(i = 1, 2, \ldots, k)$ being not all zero. For the state $|\psi_j\rangle$ $(j = 1, 2, \ldots, k)$, there exists

$$^{(n-1)\otimes}\langle\psi_j| \sum_i c_i |\psi_i\rangle^{\otimes n} = \sum_i c_i \langle\psi_j|\psi_i\rangle^{(n-1)} |\psi_i\rangle = 0, \tag{A.2}$$

Since not all $\langle\psi_j|\psi_i\rangle$ are zero, we obtain that the set $S_1 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly dependent from equation (A.2); however, it is in contradiction with the fact that the set $S_1 = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ is linearly independent. Therefore, the set $S = \{|\psi_1\rangle^{\otimes n}, |\psi_2\rangle^{\otimes n}, \ldots, |\psi_k\rangle^{\otimes n}\}$ is linearly independent. □

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